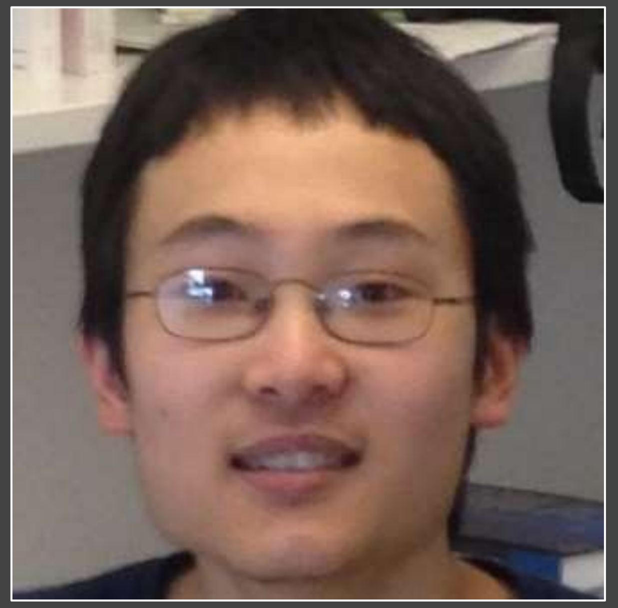
Set Cover: Two (new) Algorithms

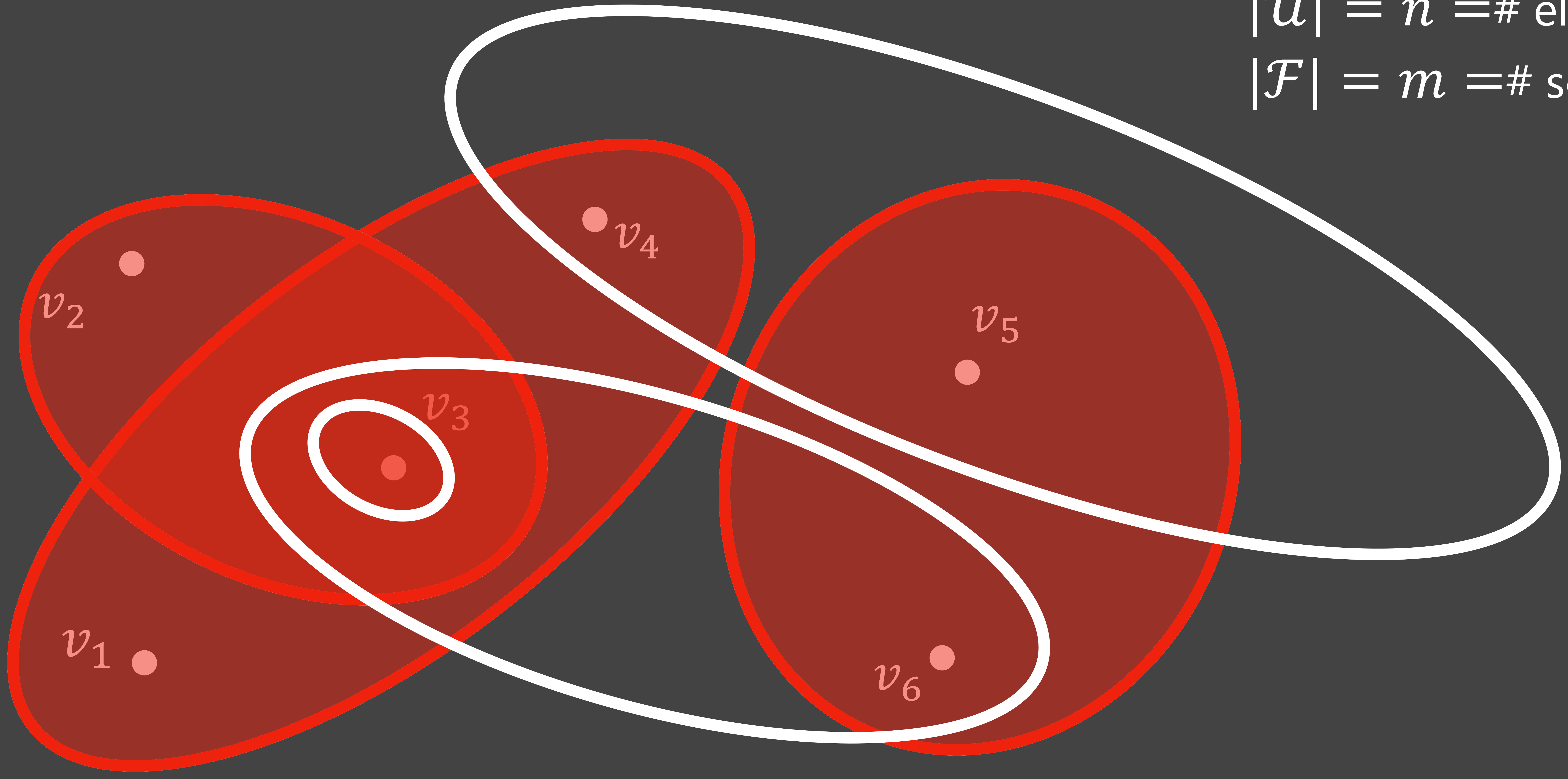

Anupam Gupta (Carnegie Mellon University)

with Greg Kehne (CMU/Harvard→?) and Roie Levin (CMU→Tel Aviv→?)
Euiwoong Lee (UMichigan) and Jason Li (CMU→Berkeley/Simons→?)

set cover

$|\mathcal{U}| = n =$# elements
$|\mathcal{F}| = m =$# sets



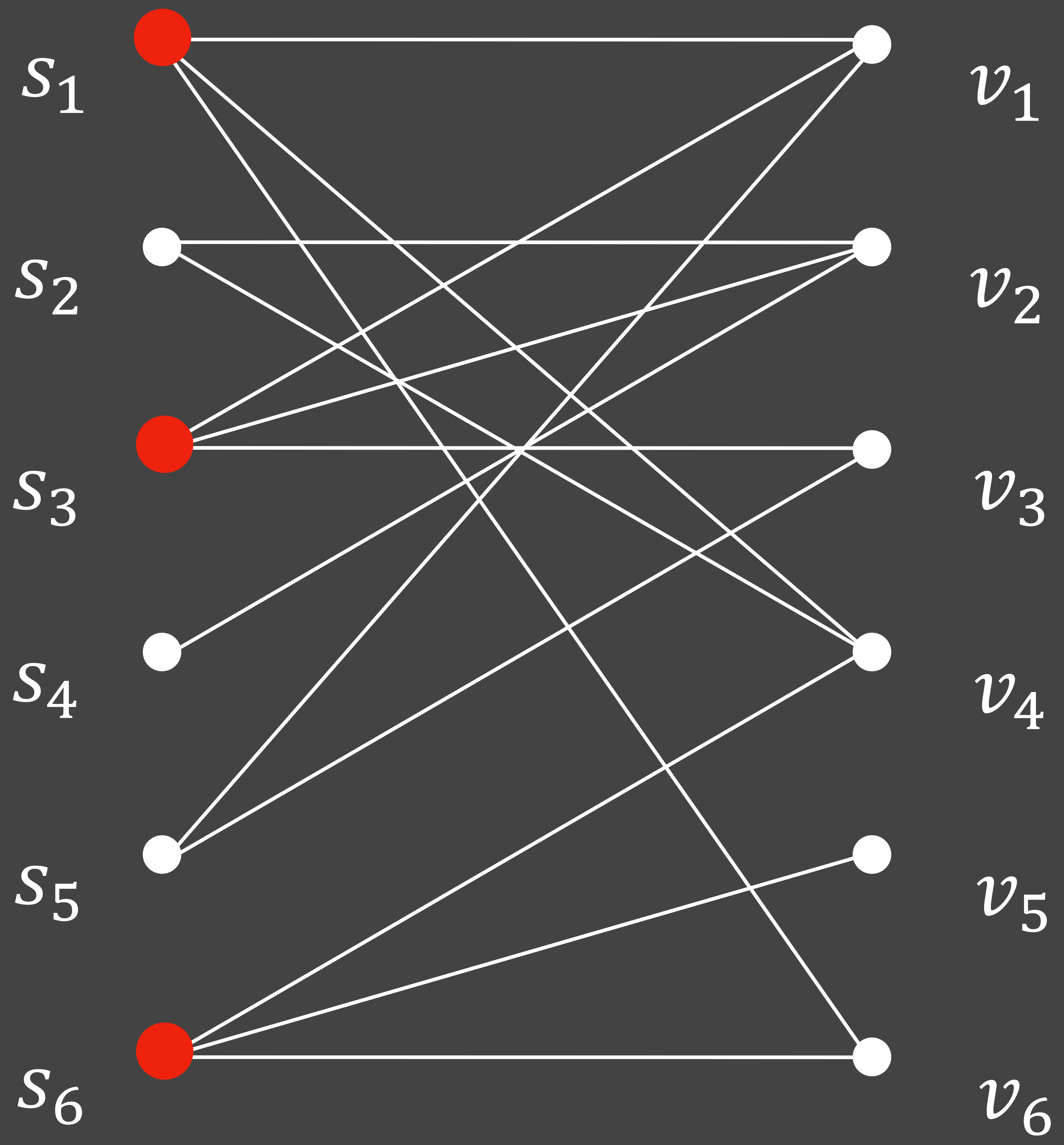
Goal: pick smallest # sets to cover all elements.

"weighted" problem: sets have costs, minimize cost of cover

set cover

$\mathcal{F}$

m sets



$\mathcal{U}$

n elements

https://theory.epfl.ch/courses/algorithmictoolbox/

Algorithmic Toolbox --- How to Solve Set Cover in x Ways

Credits	2
Lecturer	Ola Svensson
Office hours	Wednesdays 14:00 - 16:00 in INJ 112
Schedule	Mondays 14-16 in INM201.

Short description

The goal of this PhD course is to give PhD students a toolbox of algorithmic techniques in order to successfully address their favorite problems. The course emphases the illustration of the main ideas of these techniques. We prefer simplicity over details and we illustrate the algorithmic techniques in the simple and clean setting of the set cover problem. The algorithmic techniques that we plan to cover include

- Greedy algorithms
- Local search algorithms
- Linear programming
  - Randomized rounding (independent, threshold, exponential clocks)
  - Duality (primal-dual algorithms, dual fitting, and the use of complementarity slackness)
- Multiplicative weight update
- Online algorithms in adversarial and random order streams (primal-dual, potential function, and projection based)

In addition, to attending the lectures, students are required to submit a project report where they apply one of the algorithmic techniques in a more complex setting.

Schedule and references

- **Lecture 1 (Monday February 27):** *Introduction. Greedy and Local Search Algorithms*

Last modified February 2023.

set cover : previous results

set system with n elements, m sets each of size at most B

"Easyness" Theorem: [Johnson 74, Stein 74, Lovasz 75, Chvatal 79]

Set cover of cost $\leq H_B \cdot OPT \leq (1 + \ln B) \cdot OPT$ in poly-time.

"Hardness" Theorem: [Lund Yannakakis 94, Feige 98, Trevisan 01, Dinur Steurer 13]

Poly-time $(\ln B - O(\ln \ln B))$-approximation implies $P \approx NP$

set cover : **two new results**

set system with n elements, m sets each of size at most B

Local Search Theorem: [Gupta Lee Li SOSA 2023]

$(H_B - {}^{1}/_{8B} + \varepsilon) \cdot OPT$ in $\text{poly}(m, n, 1/\varepsilon)$ time

Improves on $H_B - {}^{1}/_{B^8}$ achieved by variant of greedy [Hassin Levin 05]

Random Order Theorem: [Gupta Kehne Levin FOCS 2021]

$O(\log\ mn) \cdot OPT$ in random order online model

Extends similar result for i.i.d. samples model

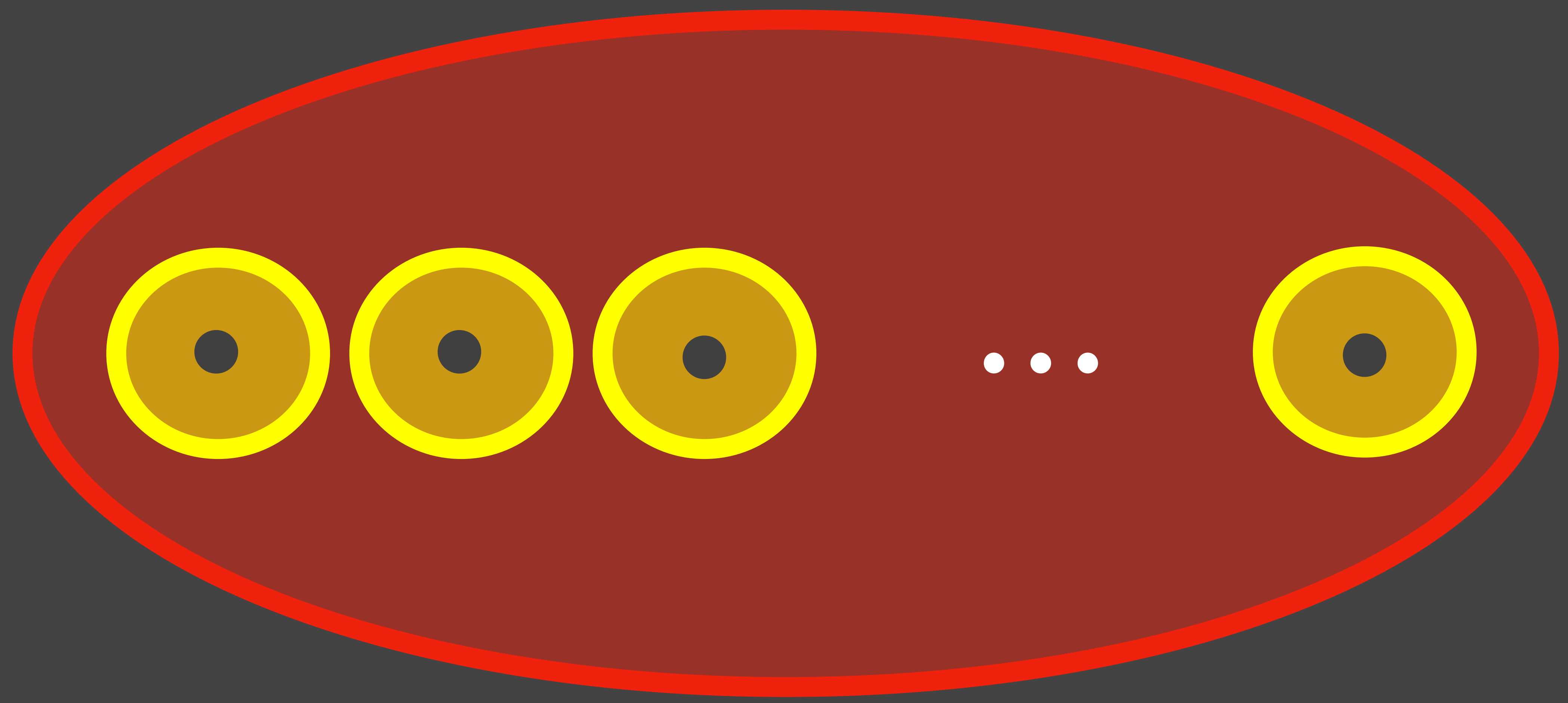
[Grandoni Gupta Leonardi Miettinen Sankowski Singh 08]

(new?) local-search algorithm

local search

Given a solution $\mathcal{S}$, perform any "local move" that improves cost $c(\mathcal{S})$

- swap $\leq c$ sets in $\mathcal{S}$ with $\leq c$ new sets; maintain coverage



unbounded
"locality gap"!

each singleton not in $\mathcal{S}$ costs 1

$\mathcal{S}$ = big set costs $M \gg n$

non-oblivious local search

Given a solution $\mathcal{S}$, perform any "local move" that improves **potential $\Phi(\mathcal{S})$**

- swap $\leq c$ sets in $\mathcal{S}$ with $\leq c$ new sets; maintain coverage

Formalized by [Khanna, Motwani, Sudan and Vazirani 98]

Useful paradigm over past decade:

Submodular maximization [Filmus Ward 14], Steiner forest [Gross et al. 18]
k-Median [Cohen-Addad+ 22], Tree Augmentation and Steiner tree [Traub Zenklusen 22]

[Rosenthal 73]

the Rosenthal potential

Solution $\mathcal{S} \subseteq \mathcal{F}$

$$\Phi(\mathcal{S}) := \sum_{S \in \mathcal{S}} c(S)\, H_{|S|}$$

Fact: $\Phi(\mathcal{S}) \geq c(\mathcal{S})$.

Fact: $\Phi(\mathcal{S}) \leq c(\mathcal{S}) \log B$ if all sets in $\mathcal{S}$ of size at most B

for simplicity…

Given set system $(E, \mathcal{F})$, define $\mathcal{F}^{\downarrow}$ to be closure by taking subsets

I.e., add in $S' \subseteq S$ for $S \in \mathcal{S}$ with cost $c(S') = c(S)$

We maintain a cover from $\mathcal{F}^{\downarrow}$ (for simplicity)

[Gupta Lee Li SOSA 2023]

our local search algorithm

$$\Phi(\mathcal{S}) := \sum_{S \in \mathcal{S}} c(S)\, H_{|S|}$$

Solution $\mathcal{S} \subseteq \mathcal{F}^{\downarrow}$

If $\mathcal{S}$ is not partition of U, drop duplicated elements, reduces potential

Add sets only from $\mathcal{F}$, so poly-time to check for move

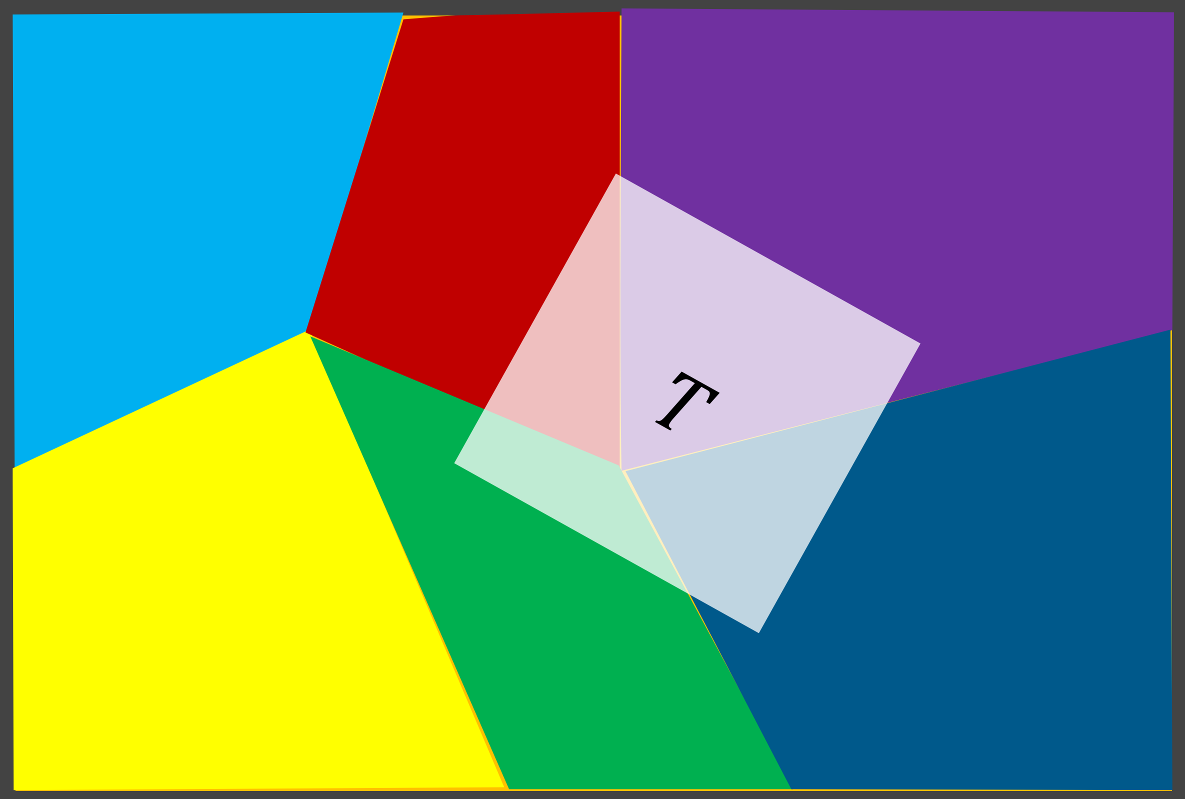
If there exists $T \in \mathcal{F}$ such that

Drop sets that are empty!

$$\mathcal{S}' := \{ S \setminus T \mid S \in \mathcal{S} \} \cup \{T\}$$

has $\Phi(\mathcal{S}') < \Phi(\mathcal{S})$, move to $\mathcal{S}'$.

$\mathcal{S}'$ also a partition



[Gupta Lee Li SOSA 2023]

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[Gupta Lee Li SOSA 2023]

local optima are good

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has $\Phi(\mathcal{S}') < \Phi(\mathcal{S})$, move to it.

Theorem: If $\mathcal{S} \subseteq \mathcal{F}^{\downarrow}$ is a local optimum,

then $c(\mathcal{S}) \leq c(\mathcal{S}^*) \cdot H_B$

Proof: For $T \in \mathcal{S}^*$,

$$0 \leq \Delta\Phi = c(T)\, H_{|T|} - \sum_{S \in \mathcal{S}} c(S) \left[H_{|S|} - H_{|S \setminus T|} \right]$$

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|S \setminus T|} + \frac{1}{|S \setminus T| + 1} + \cdots + \frac{1}{|S|} \right)$$

$|S \cap T|$ terms at least $\frac{1}{|S|}$

[Gupta Lee Li SOSA 2023]

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Sum over $T \in \mathcal{S}^*$

$$0 \le \Phi(\mathcal{S}^*) - \sum_{S \in \mathcal{S}} c(S) \sum_{T \in \mathcal{S}^*} \frac{|S \cap T|}{|S|}$$

(≥ 1)

$$\Rightarrow c(\mathcal{S}) \le \Phi(\mathcal{S}^*) \le c(\mathcal{S}^*) \cdot H_B$$

∎

extensions

Theorem: Can find solution $c(\mathcal{S}) \leq OPT \cdot (H_B + \varepsilon)$ in **poly-time**

Extension: add **two** sets at a time. $\left(H_B - {}^{1}/_{B^2} + \varepsilon\right) \cdot OPT$ via careful analysis

$(H_B - {}^{1}/_{8B} + \varepsilon) \cdot OPT$ via refined potential

Extension: add **B sets** at a time. $\left(H_B - {}^{\log B}/_{B^2} + \varepsilon\right) \cdot OPT$

Can we get $H_B - \Omega(1)$? $H_B - \omega(1)$?

today's plan

new local search algorithm

new algorithm for set cover in the random order online model

Online Set Cover

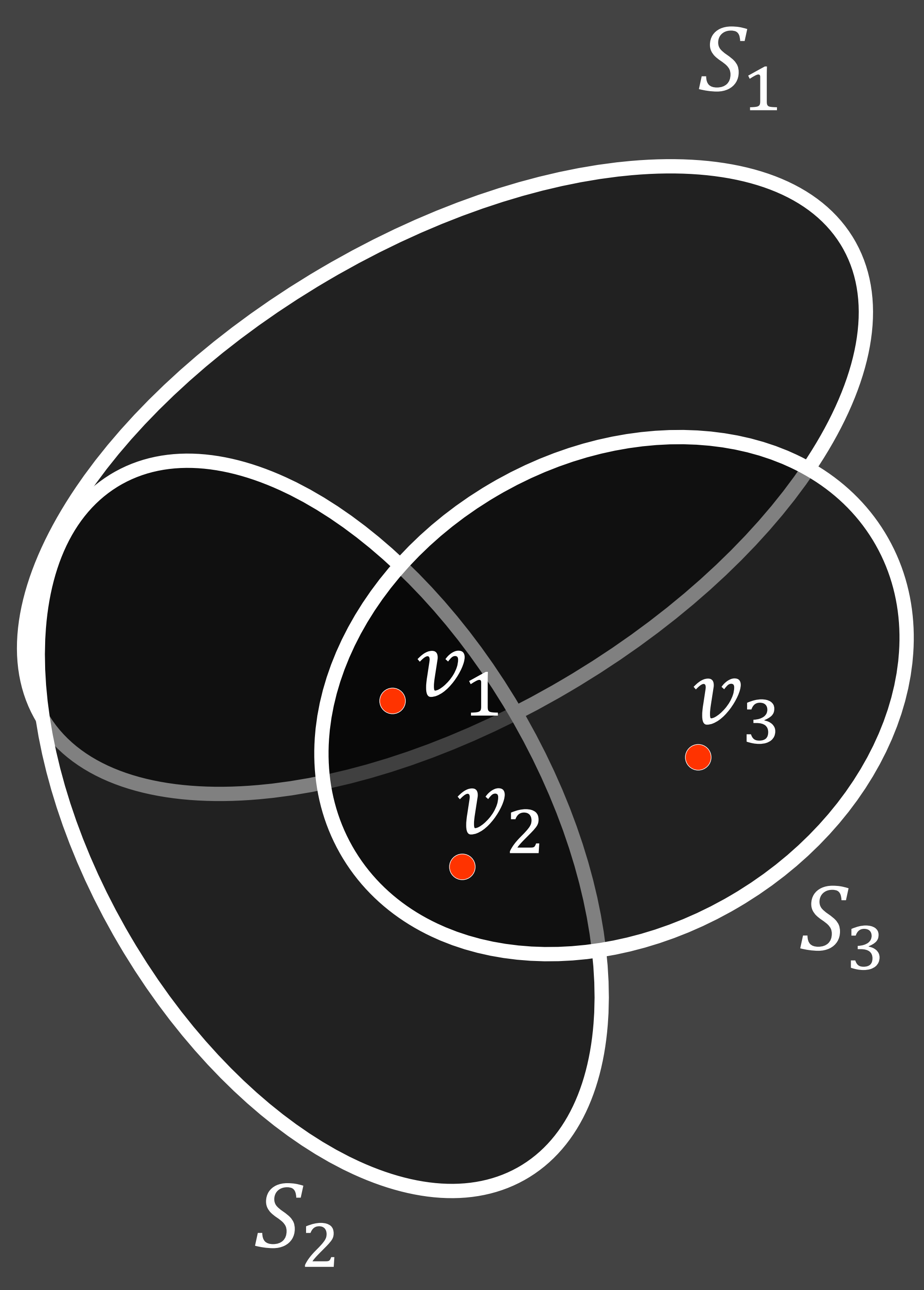
Set system. n elements arrive over time, want to maintain a cover.

Goal: minimize cost of sets picked

Competitive ratio of algorithm A:

$$\max_{\text{instances } I} \frac{\text{cost of algorithm } A \text{ on instance } I}{\text{optimal cost to serve } I}$$

Want to minimize the competitive ratio.

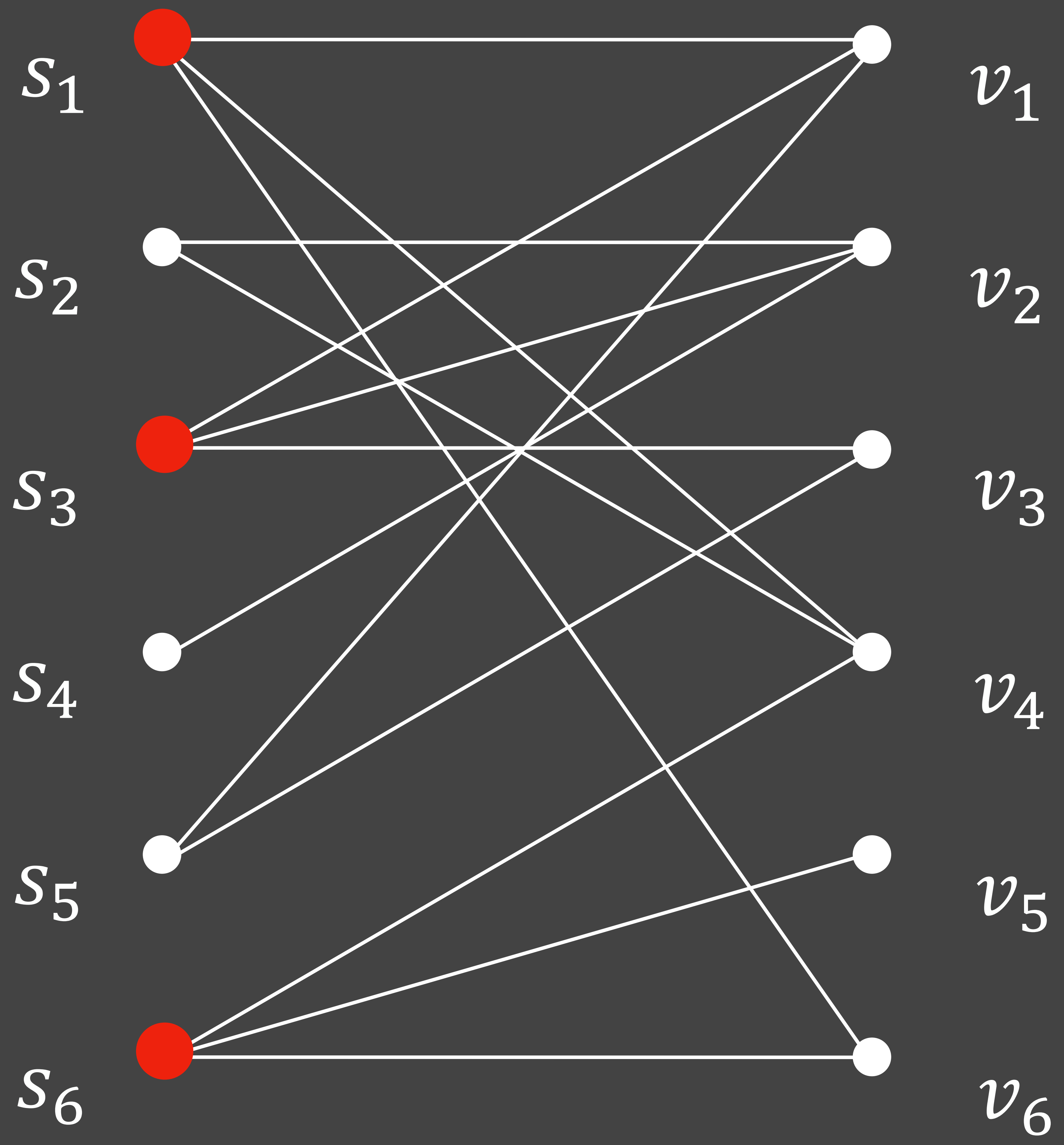


[Alon Awerbuch Azar Buchbinder Naor 03]

Online Set Cover

$\mathcal{F}$

m sets



$\mathcal{U}$

n elements

[Alon Awerbuch Azar Buchbinder Naor 03, Feige Korman 05]

Online Set Cover

Algorithm:
$O(\log n \log m)$
competitive

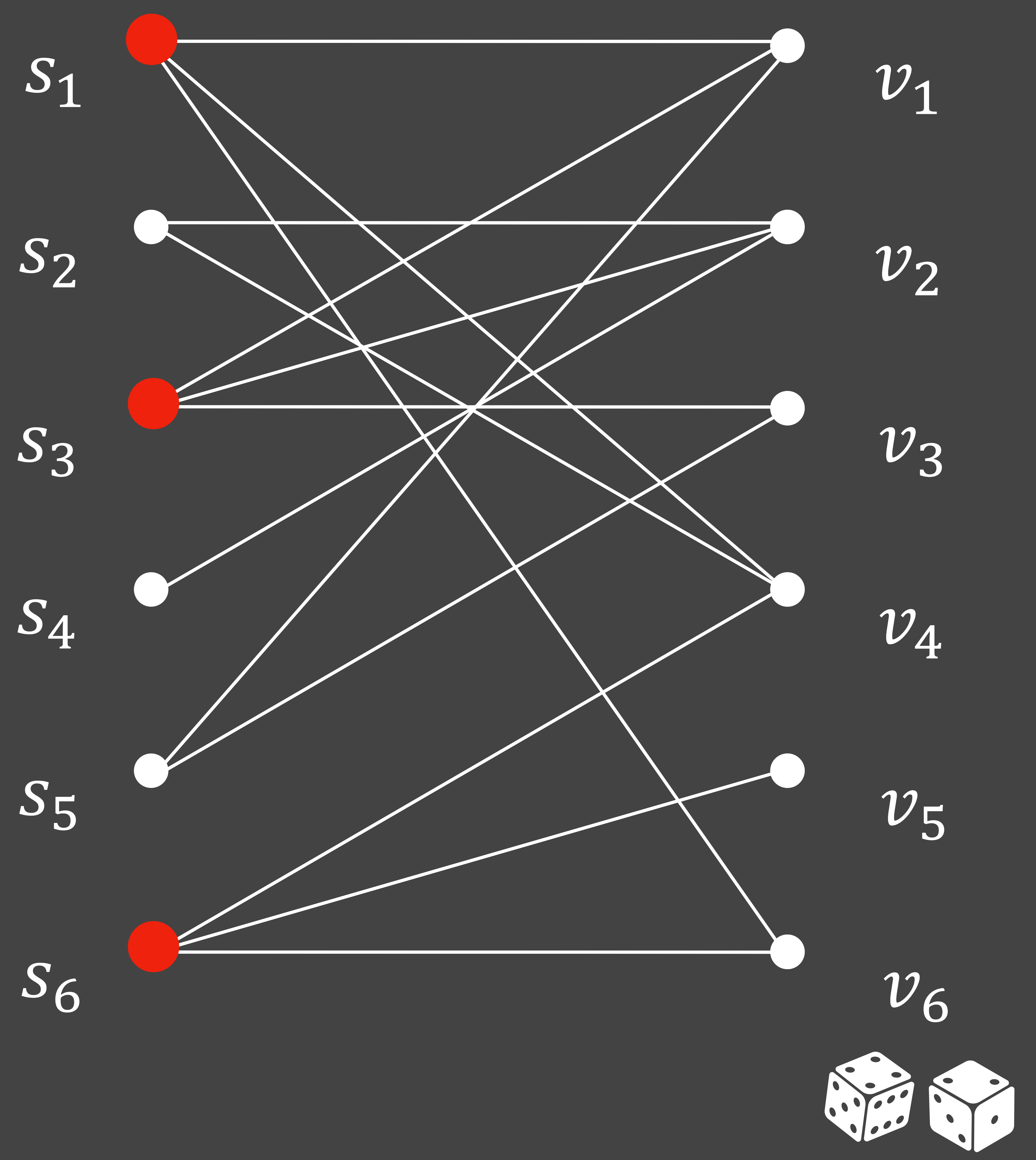
CR: $\Omega(\log n \log m)$
for deterministic algos
and for poly-time algos

Q: What happens beyond the worst case?

Random Order (RO)

$\mathcal{F}$

m sets



$\mathcal{U}$

n elements

[Gupta Kehne Levin FOCS 21]

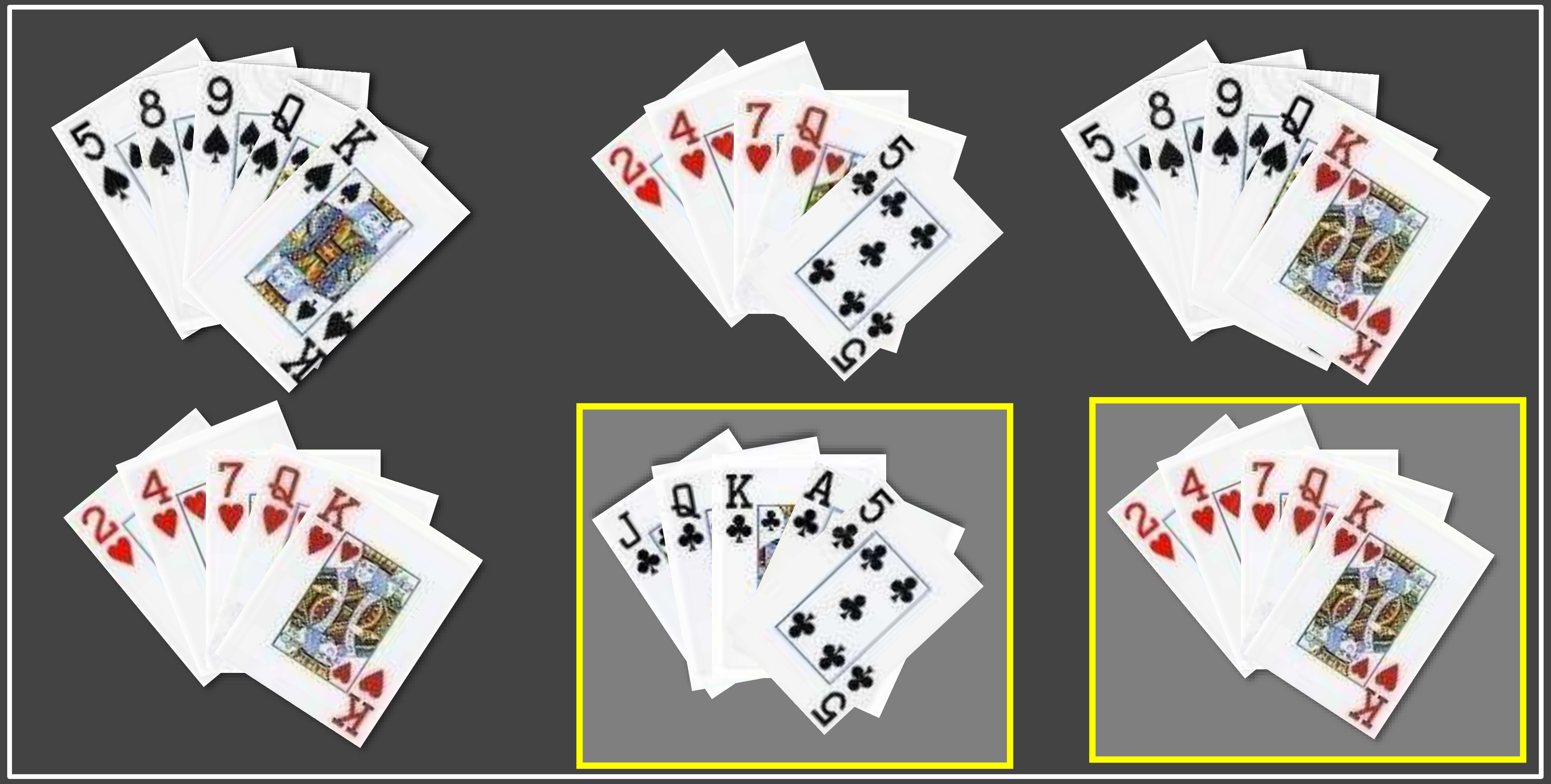
LearnOrCover

(Unit cost, exp time)

when random element v arrives
- if v not already covered, in parallel:
  1. select random remaining hand
     pick random set from it
  2. remove "hands" that don't cover v

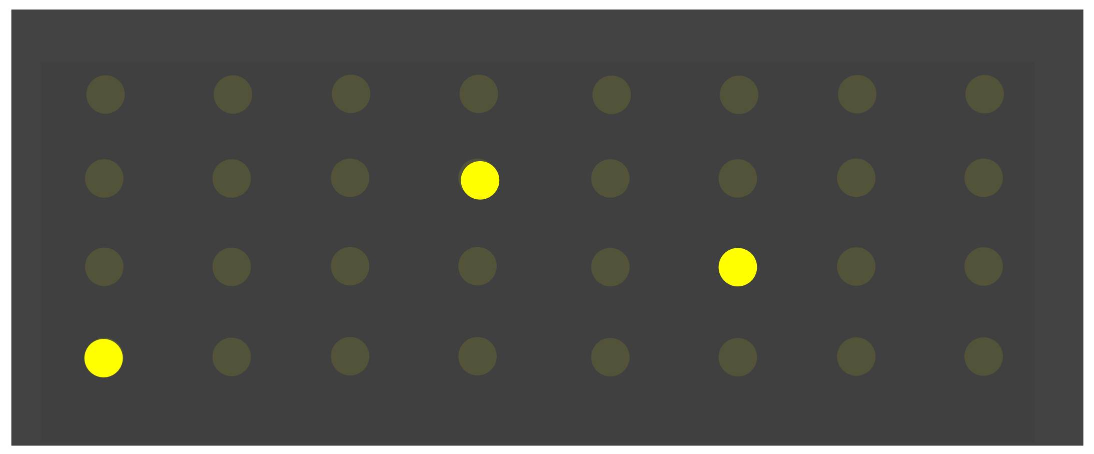
  pick any set covering v

"hands" of possible solutions



$\mathcal{P}$

$\binom{m}{k}$



$\mathcal{U}$

Q: do ½ of remaining hands cover ½ of uncovered elements?

Yes: random set covers many uncovered elements!

No: random element removes many hands!!

Sol R:

Case 1: $\geq 1/2$ of $P \in \mathcal{P}$ cover $\geq 1/2$ of $\mathcal{U}$.

R covers $\frac{|\mathcal{U}|}{4k}$ in expectation.

$\mathcal{U}$ shrinks by $\left(1 - \frac{1}{4k}\right)$ in expectation.

$|\mathcal{U}|$ initially n

$\Rightarrow \quad O(k \log n)$ COVER steps suffice.

Case 2: $> 1/2$ of $P \in \mathcal{P}$ cover $< 1/2$ of $\mathcal{U}$.

$\geq 1/2$ of $P \in \mathcal{P}$ pruned w.p. $1/2$.

$\mathcal{P}$ shrinks by $3/4$ in expectation.

$|\mathcal{P}|$ initially $\binom{m}{k} \approx m^k$

$\Rightarrow \quad O(k \log m)$ LEARN steps suffice.

$\Rightarrow O(k \log mn)$ steps suffice.

[Gupta Kehne Levin FOCS 21]

LearnOrCover
(Unit cost, exp time)

Case 1: (COVER)

$\mathcal{U}$ shrinks by $\left(1 - \frac{1}{4k}\right)$ in expectation.

Case 2: (LEARN)

$\mathcal{P}$ shrinks by $3/4$ in expectation.

$$\Phi = \frac{1}{k} \log|\mathcal{P}| + \log|\mathcal{U}|$$

Claim 1: $\Phi(0) = O(\log mn)$ and $\Phi(t) \geq 0$.

Claim 2: If v uncovered, then $E[\Delta\Phi] \leq -\Omega\left(\frac{1}{k}\right)$.

How to make polytime?

Can we reuse LEARN/COVER intuition?

[Gupta Kehne Levin FOCS 21]

LearnOrCover

(Unit cost)

Init. $x \leftarrow 1/m$.

@ time t, element v arrives:

If v covered, do nothing.

Else:

(I) Buy random $R \sim x$.

(II) $\forall S \ni v$, set $x_S \leftarrow e \cdot x_S$.

Renormalize $x \leftarrow x / \| x \|_1$.

Buy arbitrary set to cover v.

$\sum_S x_S^* \log \frac{x_S^*}{x_S^t}$

Idea: Measure convergence with potential function

$$\Phi(t) = c_1\, KL(x^* || x^t) + c_2 \log|\mathcal{U}^t|$$

$\mathcal{U}^t$:= uncovered elements @ time t

x^* := uniform distribution on OPT

Claim 1: $\Phi(0) = O(\log mn)$, and $\Phi(t) \geq 0$.

Claim 2: If v uncovered, then $E[\Delta\Phi] \leq -\frac{1}{k}$.

If $\mathbb{E}_v[\, x_v \,] > \frac{1}{4} \Rightarrow \mathbb{E}_R[k\, \Delta \log|\mathcal{U}^t|]$ drops by $\Omega(1)$.

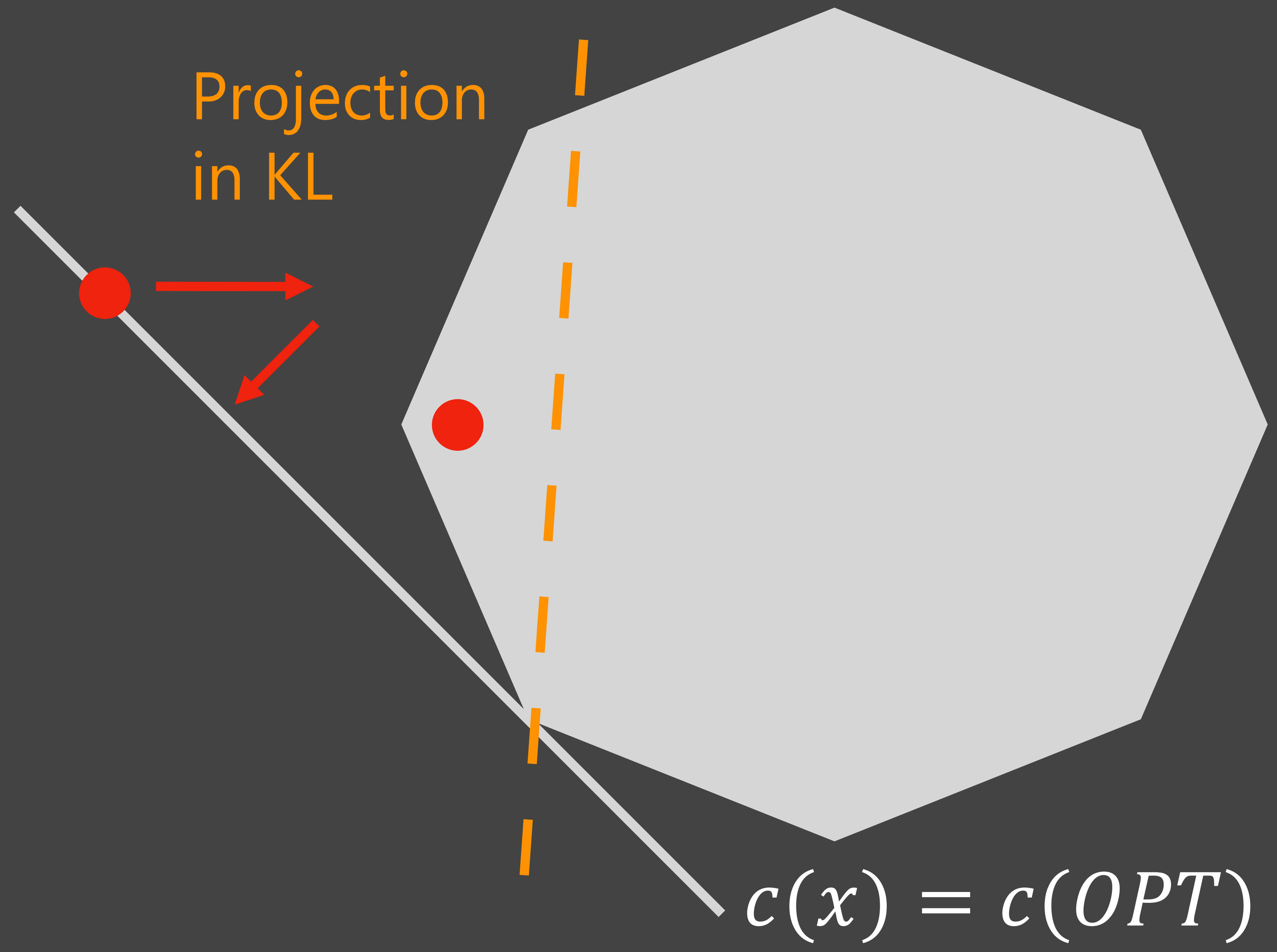
Else $\mathbb{E}_v[k\, \Delta KL]$ drops by $\Omega(1)$.

(Recall $k = |OPT|$)

LearnOrCover

(Some philosophy)

Perspective 1:



[Alon+ 03]
LearnOrCover
[Buchbinder G. Molinaro Naor 19]

Perspective 2:

Define

$$f(x) := \sum_v \max\left(0, 1 - \sum_{S \ni v} x_S\right)$$

(Goal is to minimize f in smallest # of steps)

$\nabla f|_S(x)$ = # uncovered elements in S

$\propto E[\mathbb{1}\{v \in S \mid v \text{ uncovered}\}]$

RO reveals stochastic gradient…

extensions

similar ideas work for:

- “prophet” model where requests drawn from known distributions
- covering LPs in random order
- non-metric facility location

Q1: Harder covering problems? Covering IPs w/ box constraints?

Q2: Unified theory? Reinterpret old RO results as LearnOrCover?

last slide

many interesting algorithms for basic problems still to be found

beyond-worst-case perspective behind these two results

- local search from focus on small B case
- LearnOrCover from focus on random order model

Q3: Close the $\ln B \pm O(\ln \ln B)$ gap for set cover?

Q4: use weaker random assumption than RO model?

Thanks!!!